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Characterizations of Uniform Distributions

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SUMMARY

In this paper, some characterizations of discrete and continuous uniform distributions are presented. These results are based on the conditional distribution of X, given X + Y, where X and Y are independent random variables.

Key words : Discrete uniform distribution, Rectangular distribution, Characterizations, Conditional distribution.

1. Introduction

Characterizations of distribution have been of considerable interest to many researchers. Since the uniform distribution is greatly used in several areas including random number generation in simulational studies and probabilistic number theory, characterizations of uniform distributions received special attention over the years. Some of these results may be found in the monographs of Galambos and Kotz [5] and Kakosyan *et al.* [8]. A number of articles dealing with characterizations of discrete and continuous uniform distributions have since appeared which are based on order statistics, record values, probabilistic conditions, etc. Interested readers may refer to the works of Arnold and Meeden [2], Driscoll [4], Shimizu and Huang [16], Terrell [18], Gupta [6], Szekely and Mori [17], Abdelhamid [1], Lin ([9], [10], [11], [12]), Nagaraja [13], Danial [3], Nair and Hitha [14], Too and Lin [19], Papathanasiou [15] and Kamps [7].

In this paper, some characterizations of both discrete and continuous uniform distributions are established. These results are based on the conditional distribution of X, given X + Y, where X and Y are independent random variables.

2. Characterizations of Discrete Uniform Distribution

Theorem 1. Let S be a set with a binary operation $*$ such that $a * b = a * c \Rightarrow b = c, \forall a, b, c \in S$. Let X and Y be independent S-valued random variables taking the values $\{ s_1, s_2, ..., s_n \}$ and $\{ s'_1, s'_2, ..., s'_n \}$

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respectively, such that $s_i * s'_i = k, \forall\ i$ for some $k \in S$. Let X assume each s_i with positive probabilities. Then, the conditional distribution of X given $X * Y = k$ is same as the unconditional distribution of X if and only if Y is uniformly distributed.

Proof.

Necessity. First of all, by writing

$$P(X = s_i \mid X * Y = k) = \frac{P(X = s_i,\ Y = s'_i)}{P(X * Y = k)}$$

$$= \frac{P(X = s_i)\ P(Y = s'_i)}{P(X * Y = k)} \quad (1)$$

and equating to the unconditional probability $P(X = s_i)$, it is clear that $P(Y = s'_i) = P(X * Y = k), \forall i$ which simply implies that Y is uniformly distributed.

Sufficiency. If Y is uniformly distributed, then from (1) we have

$$1 = \sum_{i=1}^{n} P(X = s_i \mid X * Y = k) = \frac{P(Y = s'_i)}{P(X * Y = k)}$$

which when used on the right-hand side of (1) reduces it to $P(X = s_i)$, proving the sufficiency.

Example 1. Taking $S = R$ and $*$ as $+$, we get :

Let X and Y be independent real-valued random variables taking the values $\{x_1, x_2, \ldots, x_n\}$ and $\{k - x_1, k - x_2, \ldots, k - x_n\}$ respectively, for some real k. Let $P(X = x_i) > 0, \forall i$. Then, Y is uniformly distributed if and only if the conditional distribution of X given $X + Y = k$ is same as the unconditional distribution of X.

Example 2. Taking $S = R - \{0\}$ and $*$ as $\times$, we get :

Let X and Y be independent real non-zero valued random variables taking the values $\{x_1, x_2, \ldots, x_n\}$ and $\left\{\frac{k}{x_1}, \frac{k}{x_2}, \ldots, \frac{k}{x_n}\right\}$ respectively, for some real non-zero k. Let $P(X = x_i) > 0, \forall i$. Then, Y is uniformly distributed if and only if the conditional distribution of X given $XY = k$ is same as the unconditional distribution of X.

Theorem 2. Suppose X and Y are independently distributed both taking the values $\{0, 1, 2, ..., n\}$ and that $P(X=r)>0, \forall r$. Then, Y is uniformly distributed if and only if

$$P(X=r \mid X+Y \leq n) = P(X=r)(\alpha-\beta r), \forall r \text{ and } \beta>0 \qquad (2)$$

Proof.

Necessity. Denoting $P(X=r)$ by p_r, $P(Y=r)$ by q_r and $P(X+Y\leq n)$ by C, one can write

$$P(X=r \mid X+Y\leq n) = \frac{P(X=r, Y\leq n-r)}{P(X+Y\leq n)}$$

$$= \frac{p_r}{C}\sum_{i=0}^{n-r} q_i \qquad (3)$$

Equating the right-hand side of (3) to $p_r(\alpha-\beta r)$, we immediately get

$$\sum_{i=0}^{n-r} q_i = C(\alpha-\beta r)$$

using which

$$q_{n-r} = \sum_{i=0}^{n-r} q_i - \sum_{i=0}^{n-r-1} q_i = C\beta, \forall r$$

which simply implies that Y is uniformly distributed.

Sufficiency. If Y is uniformly distributed, then the right-hand side of (3) is

$$\frac{p_r}{C}\frac{(n-r+1)}{(n+1)}$$

which, when noted to be of the form $p_r(\alpha-\beta_r)$ with $\beta>0$, implies the sufficiency.

Theorem 3. Suppose X and Y are independently distributed both taking the values $\{0, 1, 2, ..., n\}$ and that $P(X=r)>0, \forall r$. Then, Y is uniformly distributed if and only if

$$P(X\leq r \mid X+Y\leq n) = P(X\leq r)\{\alpha-\beta E(X \mid X\leq r)\}, \forall r \text{ and } \beta>0 \qquad (4)$$

Proof.

Necessity. With the notations used in Theorem 2, one can write

$$P(X \leq r \mid X + Y \leq n) = \frac{1}{C} \sum_{i=0}^{r} p_i \sum_{j=0}^{n-i} q_j \tag{5}$$

Similarly, write

$$P(X \leq r)\ \{\alpha - \beta E(X|X \leq r)\} = \alpha \sum_{i=0}^{r} p_i - \beta \sum_{i=0}^{r} i\, p_i \tag{6}$$

Equating the right-hand sides of (5) and (6) and subtracting the results for r – 1, we get

$$\frac{1}{C}\, p_r \sum_{j=0}^{n-r} q_j = \alpha\, p_r - \beta r p_r$$

which, after cancelling p_r on both sides and subtracting the result for r + 1, gives q_{n-r} to be a constant, $\forall r$. Hence, the necessity.

Sufficiency. If Y is uniformly distributed, the right-hand side of (5) becomes

$$\frac{1}{C} \sum_{i=0}^{r} p_i \left[\frac{n-i+1}{n+1}\right] = \frac{1}{C} \sum_{i=0}^{r} p_i - \frac{1}{C(n+1)} \sum_{i=0}^{r} i\, p_i$$

which, when noted to be of the form of the right-hand side of (6), implies the sufficiency.

3. *Characterizations of Continuous Uniform (Rectangular) Distribution*

In this section, we present some characterizations of rectangular distribution that are analogues of the results derived in the last section for the discrete uniform distribution.

Theorem 4. Suppose X and Y are independently distributed over (0, a) and are absolutely continuous with the former having a non- vanishing density. Then, Y has a rectangular distribution if and only if the conditional distribution of X given X + Y = a is same as the unconditional distribution of X.

Proof.

Necessity. Using a suggestive notation, clearly

$$f_{X \mid X+Y=a}(x) = \frac{f_X(x)\, f_Y(a-x)}{f_{X+Y}(a)} \tag{7}$$

If the right-hand side of (7) is equal to the unconditional density of X, viz., $f_X(x)$, we immediately get

$$f_Y(a-x) = f_{X+Y}(a), \ \forall x$$

which simply implies that Y has a rectangular distribution over (0, a).

Sufficiency. If Y is uniformly distributed over (0,a), then from (7) we have

$$1 = \int_0^a f_{X \mid X+Y=a}(x)\, dx = \frac{1}{a\, f_{X+Y}(a)}$$

which when used on the right-hand side of (7) reduces it to $f_X(x)$, proving the sufficiency.

Proceeding similarly, one can prove the following theorem which is clearly an analogue of Theorem 1 for the case of rectangular distribution.

Theorem 5. Let * be a binary operation on a subset S of R such that $a * b = a * c \Rightarrow b = c, \forall\, a, b, c \in S$, and further if $a \in I$, an interval of R in S, then $\{ b \in S \mid a \in I, a * b = k \}$ is also an interval of R in S. Let X and Y be independently distributed with X taking values in an interval I in S with a non-vanishing density and Y taking values in an interval J given by $J = \{ y \in S \mid x \in I, x * y = k \}$. Then, Y has a rectangular distribution over J in S if and only if the conditional distribution of X given $X * Y = k$ is same as the unconditional distribution of X.

Remark. We may note that the above theorem may be stated with R replaced by any linear space in which intervals can be defined.

Theorem 6. Suppose X and Y are independently distributed over (0, a) and are absolutely continuous with the former having a non-vanishing density. Then, Y has a rectangular distribution if and only if

$$P(X \le x \mid X+Y \le a) = P(X \le x)\ \{ \alpha - \beta\, E(X \mid X \le x) \},\ \forall\, x \text{ and } \beta > 0 \tag{8}$$

Proof.

Necessity. First of all, one can write

$$P(X \le x \mid X+Y \le a) = \int_0^x F_Y(a-u)\, dF_X(u) \Big/ \int_0^a F_Y(a-u)\, dF_X(u) \tag{9}$$

Similarly, write

$$P(X \leq x)\ \{\alpha - \beta\, E(X \mid X \leq x)\} = \int_0^x (a - \beta u)\ dF_x(u) \tag{10}$$

Denoting $\int_0^a F_Y(a-u)\, dF_X(u)$ by C, equating the right-hand sides of (9) and (10) gives

$$\int_0^x \{F_Y(a-u) - C(\alpha - \beta u)\}\, dF_X(u) = 0,\ \forall x$$

which, when differentiated both sides with respect to x, yields

$$\{F_Y(a-x) - C(\alpha - \beta x)\}\, f_X(x) = 0, \forall x$$

Since $f_X(x)$ is non-vanishing, the necessity follows.

Sufficiency. If Y is uniformly distributed, $F_Y(a-u) = \frac{a-u}{a}$ and hence the right-hand side of (9) becomes

$$\frac{\int_0^x (a-u)\ dF_X(u)}{\int_0^a (a-u)\ dF_X(u)} = \frac{aF_X(x) - F_X(x)\ E(X \mid X \leq x)}{a - E(X)}$$

which, when noted to be of the form of the right-hand side of (8), implies the sufficiency.

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REFERENCES

[1] Abdelhamid, S.N., 1985. On a characterization of rectangular distribution. *Statist. Probab. Lett.*, **3**, 235-238.

[2] Arnold, B.C. and Meeden, G., 1976. A characterization of the uniform distribution based on summation modulo one, with application to fractional backlogs. *Austr. J. Statist.*, **18**, 173-175.

[3] Danial, E.J., 1989. Characterization of the special discrete distribution through infinite, non-infinite and finite divisibility. *Statist. Probab. Lett.*, **8**, 1-7.

[4] Driscoll, M.F., 1978. On pairwise and mutual independence : characterization of rectangular distribution. *J. Amer. Statist. Assoc.*, **73**, 432-433.

[5] Galambos, J. and Kotz, S., 1978. *Characterization of probability distributions.* Lecture Notes in Mathematics, No. 675, Springer-Verlag, New York.

[6] Gupta, R.C., 1984. Relationship between order statistics and record values and some characterization results. *J. Appl. Probab.*, **21**, 425-430.

[7] Kamps, U., 1991. Inequalities for moments of order statistics and characterizations of distributions. *J. Statist. Plann. Inf.*, **27**, 397-404.

[8] Kakosyan, A.V., Klebanov, L.B. and Melamed, J.A., 1984. *Characterization of distributions by the method of intensively monotone operators.* Lecture Notes in Mathematics, No. 1088, Springer-Verlag, New York.

[9] Lin, G.D., 1987. On characterizations of distributions via moments of record values. *Probab. Theor. Rel. Fields*, **74**, 479-483.

[10] Lin, G.D., 1988a. Characterizations of uniform distribution and of exponential distribution. *Sankhya,* **A50**, 64-69.

[11] Lin, G.D., 1988b. Characterizations of distributions via relationships between two moments of order statistics. *J. Statist. Plann. Inf.*, **19**, 73-80.

[12] Lin, G.D., 1989. The product moments of order statistics with applications to characterizations of distributions. *J. Statist. Plann. Inf.*, **21**, 395-406.

[13] Nagaraja, H.N., 1988. Some characterizations of discrete distributions based on linear regressions of adjacent order statistics. *J. Statist. Plann. Inf.*, **20**, 65-75.

[14] Nair, N.U. and Hitha, N., 1989. Characterization of discrete models by distributions based on their partial sums. *Statist. Probab. Lett.*, **8**, 335-337.

[15] Papathanasiou, V., 1990. Some characterizations of distributions based on order statistics. *Statist. Probab. Lett.*, **9**, 145-147.

[16] Shimizu, R. and Huang, J.S., 1983. On a characteristic property of the uniform distribution. *Ann. Inst. Statist. Math.*, **35**, 91-94.

[17] Szekely, G.J. and Mori, T.F., 1985. An extremal property of rectangular distributions. *Statist. Probab. Lett.*, **3**, 107-109.

[18] Terrell, G.R., 1983. A characterization of rectangular distribution. *Ann. Probab.*, **11**, 823-826.

[19] Too, Y.H. and Lin, G.D., 1989. Characterizations of uniform and exponential distributions. *Statist. Probab. Lett.*, **7**, 357-359.